Learning to be Risk Averse?

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Roadmap.

Abstract:

The purpose of this research is to search for the best (highest performing) risk profile of agents who successively choose among risky prospects. An agent's risk profile is his attitude to perceived risk, which can vary from risk preferring to risk neutral (an expected-value decision maker) to risk averse.

We use the Genetic Algorithm to search in the complex stochastic space of repeated lotteries. We find that agents with a CARA utility function learn to possess risk-neutral risk profiles. Since CARA utility functions are wealth-independent, this is not surprising. When agents have wealth-dependent, CRRA utility functions, however, they also learn to possess risk profiles that are about risk neutral (from slightly risk-averse to even slightly risk-preferring), which is surprising.

1. Introduction

Informally, it is widely held that in an uncertain world, with the possibility of the discontinuity of bankruptcy, the most prudent risk profile is risk aversion. Indeed, "Risk aversion is one of the most basic assumptions underlying economic behavior" (Szpiro 1997), perhaps because "a dollar that helps us avoid poverty is more valuable than a dollar that helps us become very rich" (Rabin 2000). But is risk aversion the best risk profile? Even with bankruptcy as a possibility?

To answer this question, we use two kinds of utility function (the wealth-independent exponential utility function, or Constant Absolute Risk Aversion CARA, and the Constant Relative Risk Aversion CRRA function, which is sensitive to the agent's level of wealth) and run computer experiments in which each agent chooses among three lotteries, and is then awarded with the outcome of the chosen lottery k.

Repetition of this choice by many agents allows us use a technique from machine learning — the Genetic Algorithm (Holland 1992) — to search for the best risk profile, where "best" means the highest average payoff when chosing among lotteries.

Modelling the agent's utility directly allows us to avoid the indirect inference of Szpiro (1997), who argues that the evolutionary learning technique of the GA does two things: it allows wealth-maximizing agents to succeed even in highly stochastic environments, and it allows the emergence of risk aversion. Indeed, Szpiro argues that risk aversion is the best risk profile to adopt in such an environment.

2. Decisions under Uncertainty and Risk Profiles

The von Neumann-Morgenstern formulation of the decision-maker's attitude to risk is based on the observation that individuals are not always expected-value decision makers. That is, there are situations in which people apparently prefer a lower certain outcome to the higher expected (or probility-weighted) outcome of an uncertain prospect (where the possible outcomes and their possibly subjective, or Bayesian, probabilities are known).

An example is paying an insurance premium that is greater than the expected loss without insurance. On the other hand, people will sometimes "gamble" by apparently preferring a lower uncertain outcome to a higher sure thing: this is risk-preferring.

We can formalise this by observing that, by definition, the utility of a lottery is its expected utility, or

$$U(\mathrm{L}) = \sum p_i U(x_i), \tag{1}$$

where each (discrete) outcome x_i occurs with probability p_i, and $U(x_i)$ is the utility of outcome x_i. It is useful to define the Certainty Equivalent $\tilde{x}$ (or C.E.), which is a certain outcome which has the identical utility as the lottery:

$$U(\tilde{x}) \equiv U(\mathrm{L}) = \sum p_i U(x_i). \tag{2}$$

We can use the C.E. to describe the decision-maker's risk profile (Howard 1968). Define the Expected Value $\bar{x}$ of the Lottery as:

$$\bar{x} = \sum p_i x_i. \tag{3}$$

When $\tilde{x} = \bar{x}$ then the decision-maker's utility function exhibits risk neutrality, when $\tilde{x} < \bar{x}$ then risk aversion, and when $\tilde{x} > \bar{x}$ then risk preference.

Approximating the Certainty Equivalent

Expand utility $U(.)$ about the expected value $\bar{x}$.

$$U(x_0) \approx U(\bar{x}) + (x_0 - \bar{x})U'(\bar{x}) + \tfrac{1}{2}(x_o - \bar{x})^2 U''(\bar{x})$$

The C. E. $\tilde{x}$ of a continuous lottery is obtained by integration over the probability density function (p.d.f.) $f_x(.)$:

$$\begin{aligned} U(\tilde{x}) &= \int dx_0 U(x_0) f_x(x_0) \\ &\approx U(\bar{x}) + 0 + \tfrac{1}{2}\sigma^2 U''(\bar{x}) \end{aligned} \qquad (4)$$

But, by expansion,

$$U(\tilde{x}) \approx U(\bar{x}) + (\tilde{x} - \bar{x})U'(\bar{x}). \qquad (5)$$

Therefore, from equations (4) and (5),

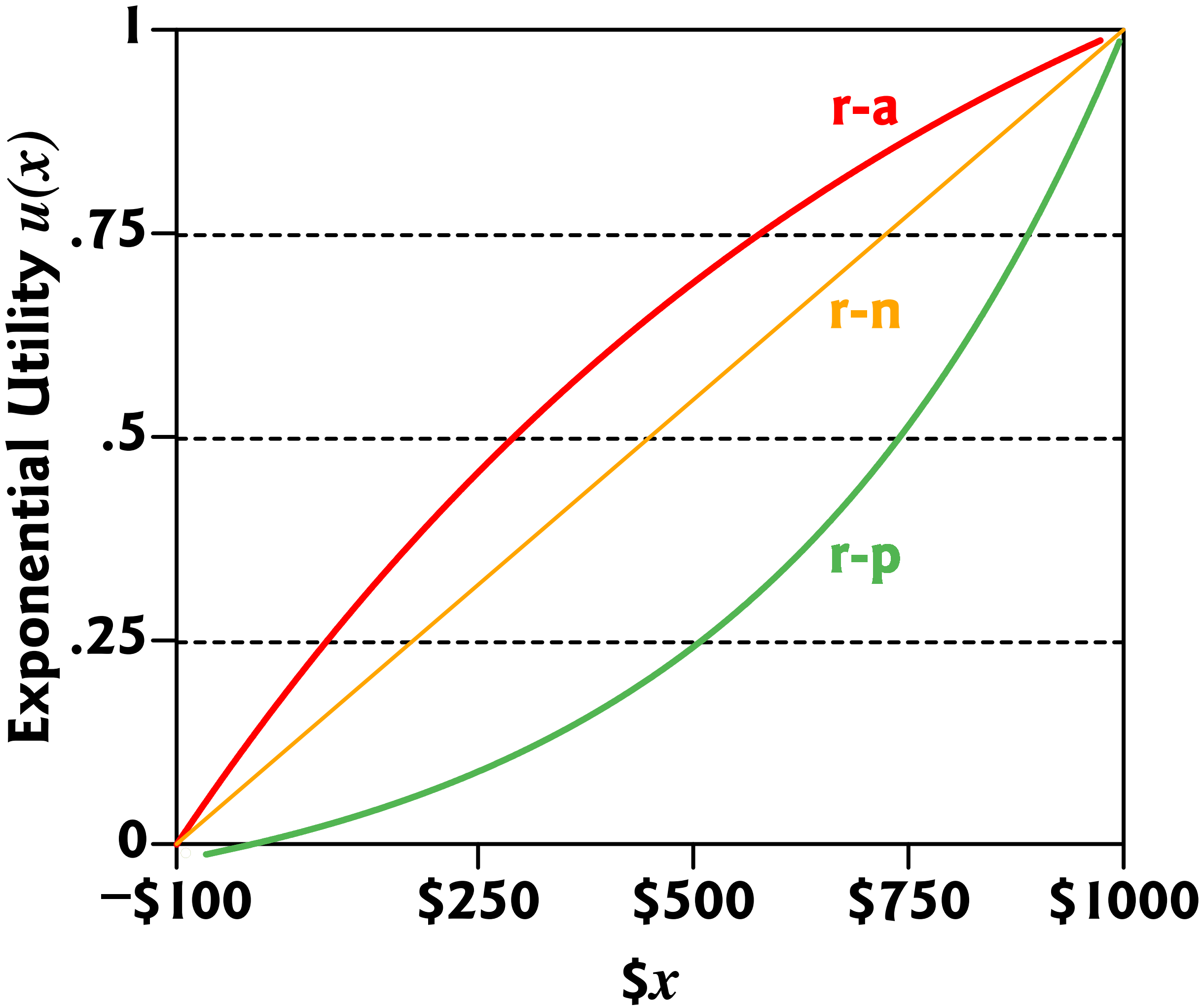
$$\begin{aligned} \tilde{x} - \bar{x} &\approx \tfrac{1}{2}\sigma^2 \frac{U''(\bar{x})}{U'(\bar{x})} \\ \therefore\ \tilde{x} &\approx \bar{x} + \tfrac{1}{2}\sigma^2 \frac{U''(\bar{x})}{U'(\bar{x})} \end{aligned} \qquad (6)$$

Risk aversion

Risk aversion is not indicated by the slope of the utility curve: it's the *curvature*: if the utility curve is locally —

1. **linear (say, at a point of inflection), then the decision maker is *locally risk neutral.***
2. **concave (its slope is decreasing — Diminishing Marginal Utility), then the decision maker is *locally risk averse;***
3. **convex (its slope is increasing), then the decision maker is *locally risk preferring.***

Risk-averse, risk-neutral, and risk-preferring exponential utility functions.



3. Utility Functions

We consider two types of utility function: those which exhibit constant risk preference across all outcomes (so-called wealth-independent utility functions, or Constant Absolute Risk Aversion CARA functions), and those where the risk preference is a function of the wealth of the decision maker (the Constant Relative Risk Aversion CRRA functions).

Wealth Independence

If an increase of all outcomes in a lottery by an amount Δ increases the C.E. by Δ, then the decision maker exhibits wealth independence:

$$U(\tilde{x} + \Delta) = U(\mathrm{L}') = \sum p_i U(x_i + \Delta).$$

Acceptance of this property restricts possible utility functions to be linear (risk neutral) or exponential, the *wealth-independent,* or constant-absolute-risk-aversion (CARA) functions.

Acceptance of wealth independence leads to the characterisation of risk preference by a single number, the *risk aversion coefficient,* γ.

Since CARA utility functions are wealth-independent, any aversion to bankruptcy is thus precluded, by definition.

3.1 CARA Utility Functions

When utility is linear in outcomes, the decision maker is risk-neutral, across all outcomes, but such a simple constant-risk-profile utility function is of no further interest. Instead, we consider the exponential CARA functions, where utility U is given by

$$U(x) = 1 - e^{-\gamma x}, \tag{7}$$

where $U(0) = 0$ and $U(\infty) = 1$, and
where γ is the *risk aversion coefficient*:

$$\gamma \equiv -\frac{U''(x)}{U'(x)}. \tag{8}$$

Risk Aversion with Exponential Utility

From equations (6) and (8), for exponential utility,

$$\tilde{x} \approx \bar{x} - \tfrac{1}{2}\sigma^2\gamma$$

which indicates that when $\gamma = 0$, then $\tilde{x} \approx \bar{x}$ (risk neutrality), when $\gamma > 0$, then $\tilde{x} < \bar{x}$ (risk averse), and when $\gamma < 0$, then $\tilde{x} > \bar{x}$ (risk preferring), with positive variance.

Summarizing this:

Sign of γ	Risk profile	Curvature
$\gamma = 0$	risk neutral	$U''(x) = 0$
$\gamma > 0$	risk averse	$U''(x) < 0$
$\gamma < 0$	risk preferring	$U''(x) > 0$

3.2 CRRA Utility Functions

We want a utility function which is *not* wealth-independent, to see whether that will result in risk-averse agents doing best.

The Arrow-Pratt measure of relative risk aversion (RRA) is defined as

$$\rho(w) = -w \, \frac{U''(w)}{U'(w)} = w\gamma \tag{9}$$

This introduces wealth w into the agent's risk preferences, so that lower wealth can be associated with higher risk aversion. Risk aversion coefficient γ is as in equation (8).

The Constant Elasticity of Substitution (CES) utility function:

$$U(w) = \frac{w^{1-\rho}}{1-\rho}, \quad w > 0, \tag{10}$$

exhibits CRRA, equation (9).

Risk Aversion with CES Utility

In the CRRA simulations, we use the cumulative sum of the realisations of payoffs won in previous lotteries chosen by the agent plus the possible payoff in this lottery as the wealth w in equation (10).

Each agent codes for ρ.

From equation (6), the C.E. with CES utility is approximated by

$$\tilde{x} \approx \bar{x} - \frac{1}{2} \frac{\rho}{w} \sigma^2$$

Iff $\frac{1}{2} \frac{\rho}{w} \sigma^2 > 0$ (or $\rho / w > 0$), then C.E. $\tilde{x}$ < expected mean $\bar{x}$, and the decision maker is risk averse.

With $w > 0$, $\rho > 0$ is equivalent to risk aversion.
With $w > 0$ and $\rho = 1$, the CES function becomes the (risk-averse) logarithmic utility function, $U(w) = \log(w)$.

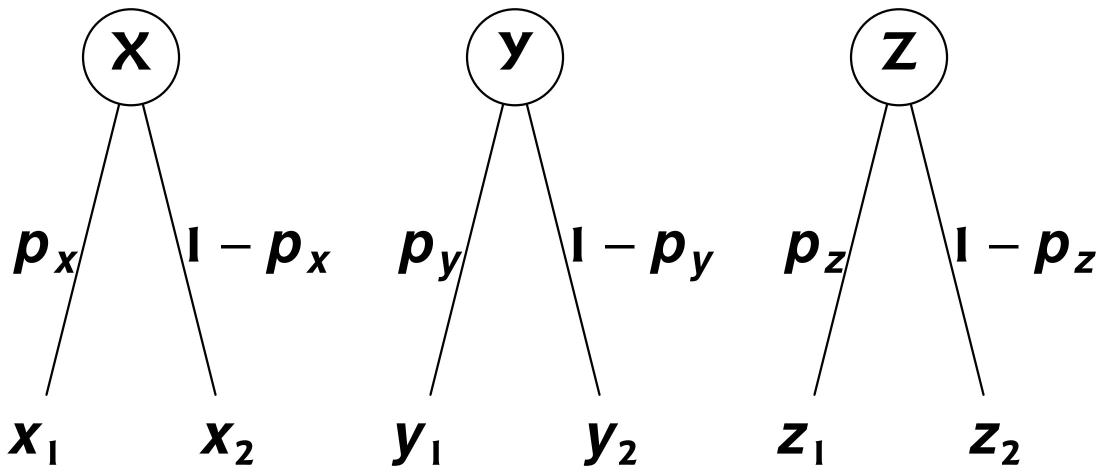
4. The Simulations

Each lottery is randomly constructed: the two payoffs ("prizes") are randomly chosen in the interval between − and + MAP, usually 100; and the probability is also chosen randomly. (Each lottery has, of course, a single degree of freedom for probability). Each agent calculates the expected utility of each of the three lotteries, using its utility function (a function of its γ or ρ/W), and chooses the lottery k with the highest expected utility. To do this, agents know the prizes and probabilities of all three lotteries.

Then the actual (simulated) outcome of the chosen lottery k is randomly realised, using its probability. The winnings of the Constant Absolute Risk Aversion agent (respectively, the wealth of the Constant Relative Risk Aversion agent) is incremented accordingly. Each agent chooses 1000 lotteries. At this stage there is a population of agents, each of which has a average winnings or a cumulative level of wealth, based on its risk profile and the successive outcomes of its choices among the lotteries.

Diagrammatically

Three two-prize lotteries with random prizes and probability:



Calculate the three expected utilities, functions of γ (or ρ and w):

$$U(X) = p_x U(x_1) + (1 - p_x) U(x_2)$$
$$U(Y) = p_y U(y_1) + (1 - p_y) U(y_2)$$
$$U(Z) = p_z U(z_1) + (1 - p_z) U(z_2)$$

Choose the lottery I with the highest expected utility. Win whichever prize (i_1 or i_2) is realised in that lottery, based in the lottery's probability p_i.

Searching with the Genetic Algorithm

We now use an implementation of the Genetic Algorithm (Gilbert 2004) to search for the best risk profile. That is, we select the best-performing agents to be the "parents" of the next generation of agents, which is generated by "crossover" and "mutation" of the chromosomes of the pairs of parents. Each of the new generation of agents chooses the lottery *k* with highest expected utility a thousand times. Again, the best are selected to be the parents of the next generation.

We use the GA simulation in this search as an empirical alternative to solving for the best (highest performing) risk profile analytically. Note that Rabin (2000) asserts that "theory actually predicts virtual risk neutrality." We return to this paper in the Discussion below.

4.1 The Simulations with CARA Utility

Using NetLogo (Wilensky 1999), we model each agent as a binary string which codes to its risk-aversion coefficient, γ, in the interval $\pm$ 1.048576.

Each lottery is a two-prize lottery, where each prize is chosen from a uniform distribution, between − and + MAP (maximum absolute prize), where MAP can be set up to 100 by the simulator, and the single probability is chosen randomly from uniform [0,1].

Each agent chooses the lottery k with the highest expected utility from equations (1) and (7), based on its value of γ. Then a realised outcome is calculated for that lottery, based on its probability.

Each agent faces 1000 lottery choices, and the cumulative winnings that agent's "fitness" for the Genetic Algorithm.

The CARA Results

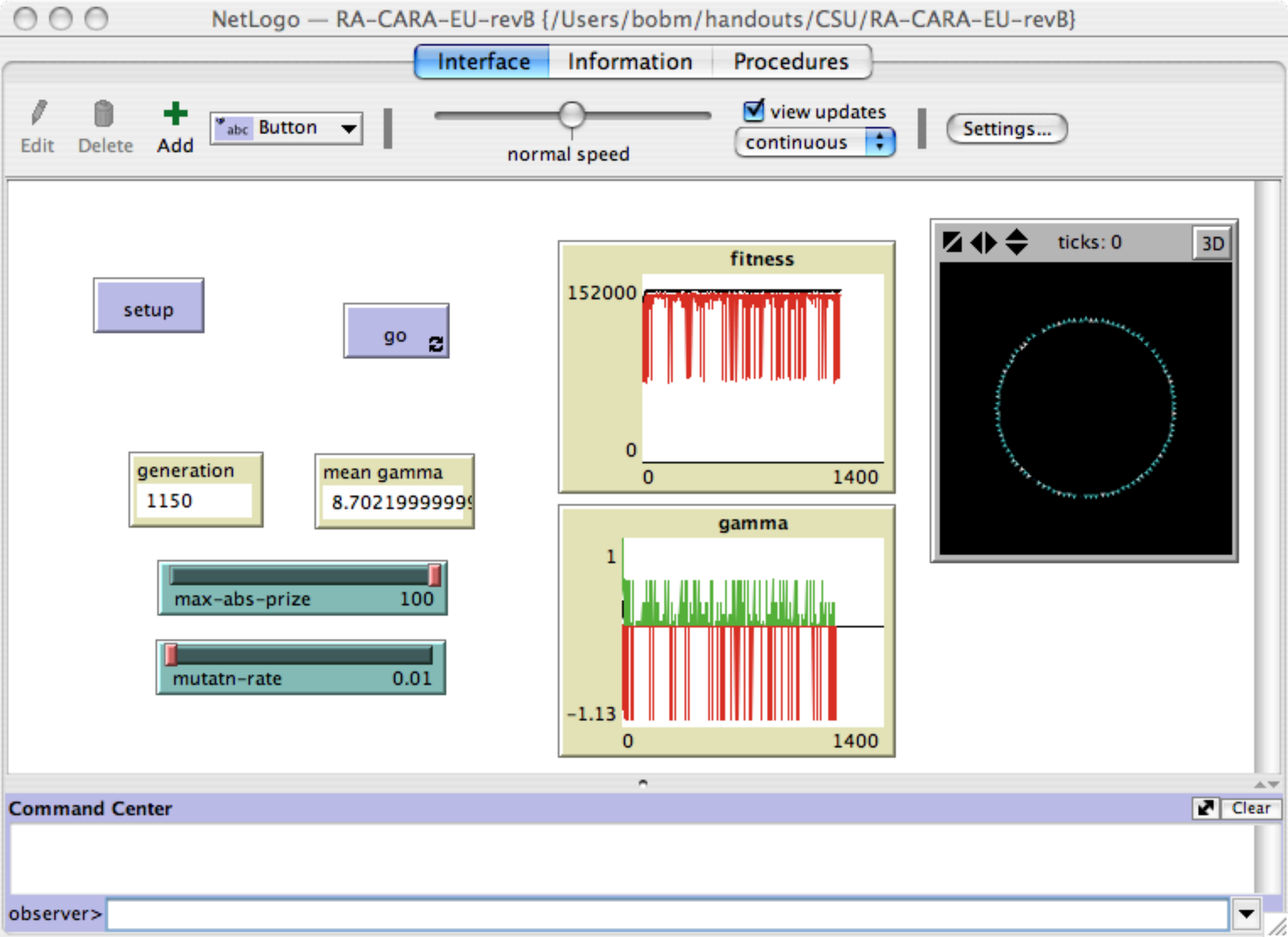
See `http://www.agsm.edu.au/bobm/teaching/SimSS/NetLogo4-models/RA-CARA-EU-312p.html` **for a Java aplet and the NetLogo code.**

The windows captured from the NetLogo simulations show three things clearly:

1. **The mean (black) fitness grows quickly to a plateau after 20 generations or so;**
2. **the mean, maximum, and minimum risk-aversion coefficients γ (resp. black, green, red) converge to close to zero (risk neutrality) over the same period, and**
3. **Any γ deviation from zero up (more risk-averse) or down (more risk-preferring) leads to the minimum (red) fitness in that generation collapsing from close to the mean fitness.**

These observations clearly show that CARA agents perform best (in terms of their lottery winnings) who are closest to risk neutral (γ = 0).

Too risk averse, and they forgo fair lotteries; too risk preferring and they choose too many risky lotteries.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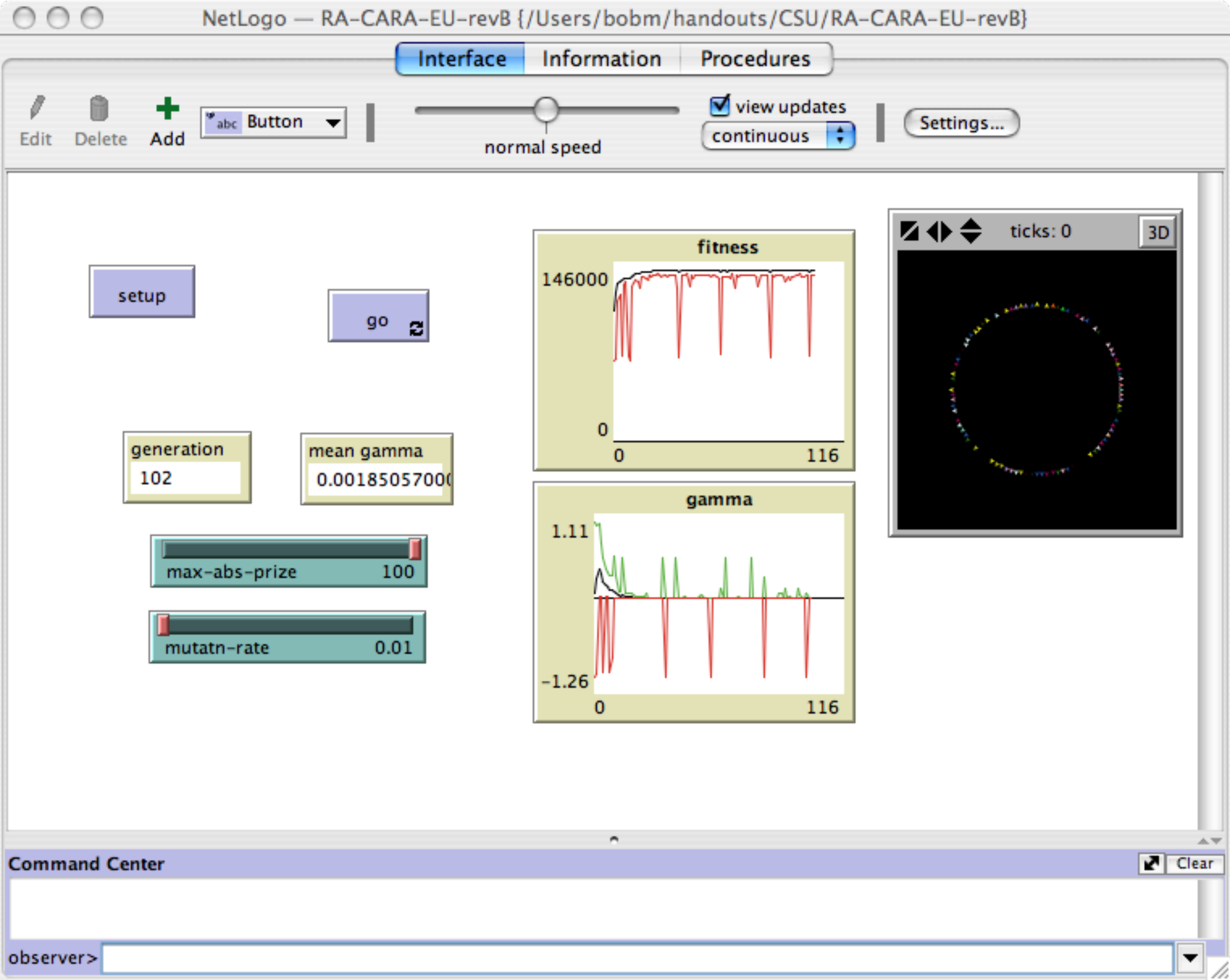
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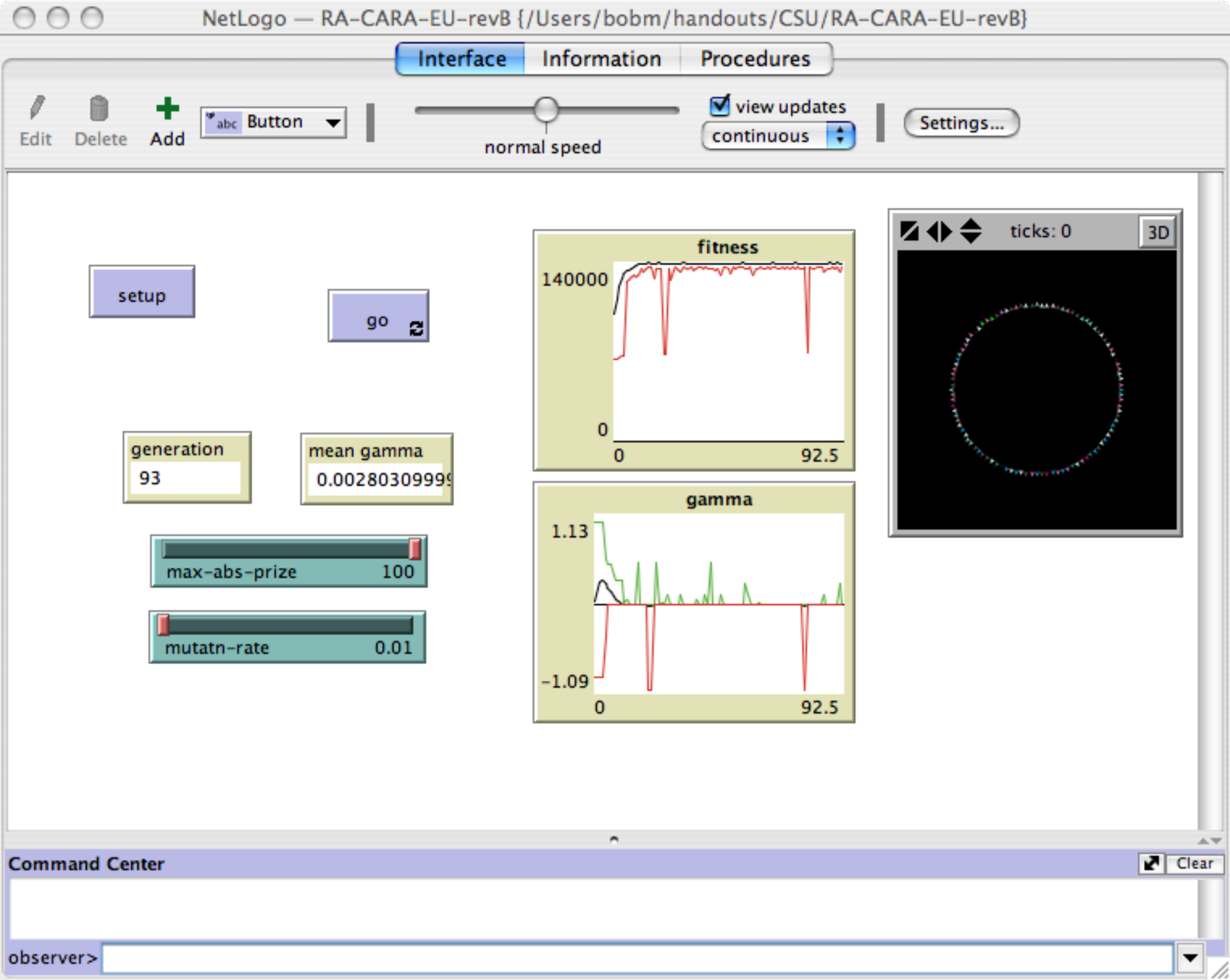

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4.2 The CRRA Results

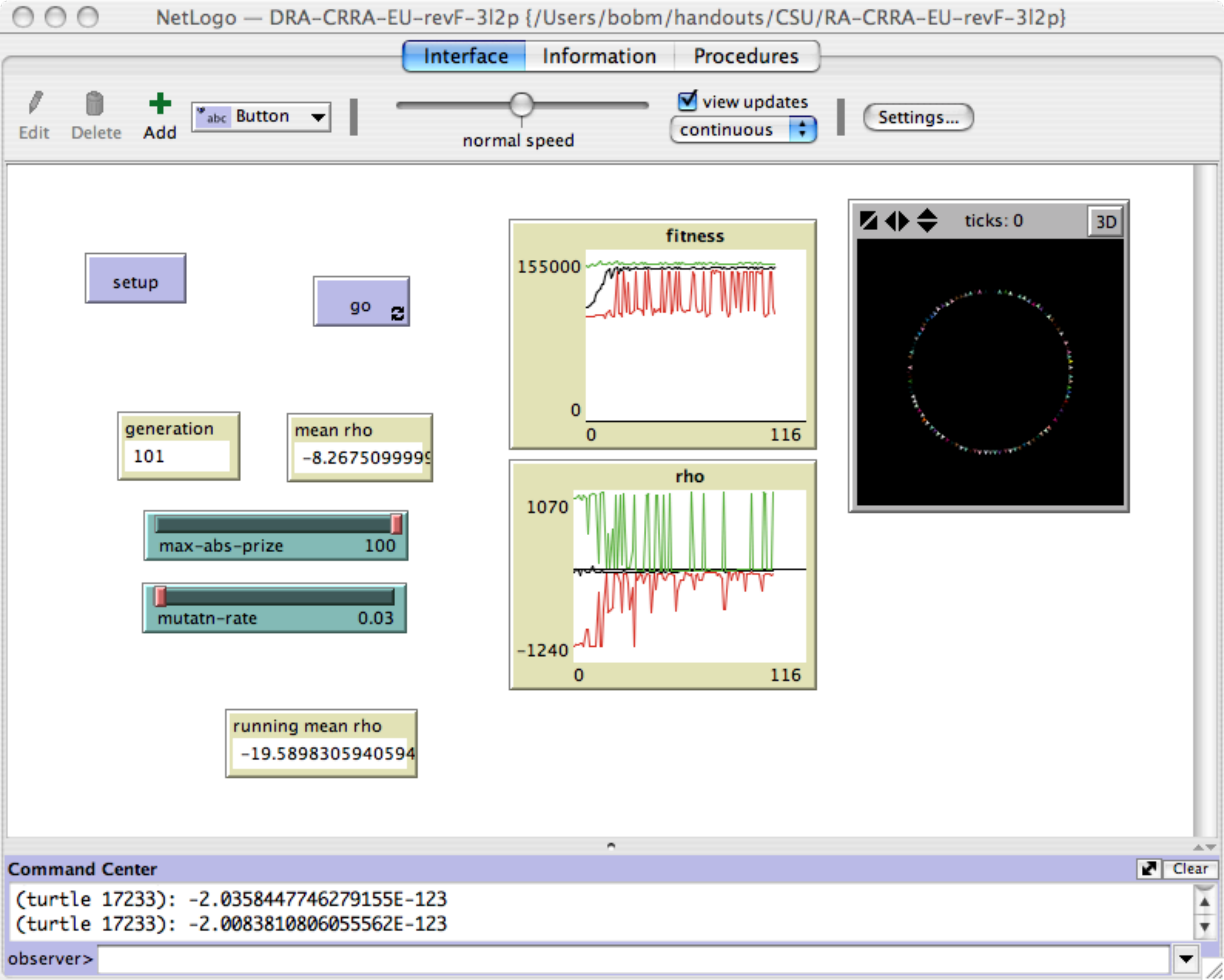
See http://www.agsm.edu.au/bobm/teaching/SimSS/NetLogo4-models/DRA-CRRA-EU-revCD-312p.html **for a Java aplet and the NetLogo code.**

Despite our prior belief, the CARA agents do not learn to be risk averse, but to be risk neutral. Is this because the wealth-independent CARA utility function precludes bankruptcy?

We could, of course, put a floor on agent wealth, below which is oblivion, but better to use a utility formulation that is not wealth independent and repeat the search. We use the CES utility functions (equation (10)) that exhibits CRRA.

The results are surprising: the CRRA agents do not learn to be risk averse, but are very close to risk neutral.

Remember: $\gamma = \dfrac{\rho}{w}$**, so dividing the** ρ **values by the high** w **values attained implies corresponding minute values of** γ **here.**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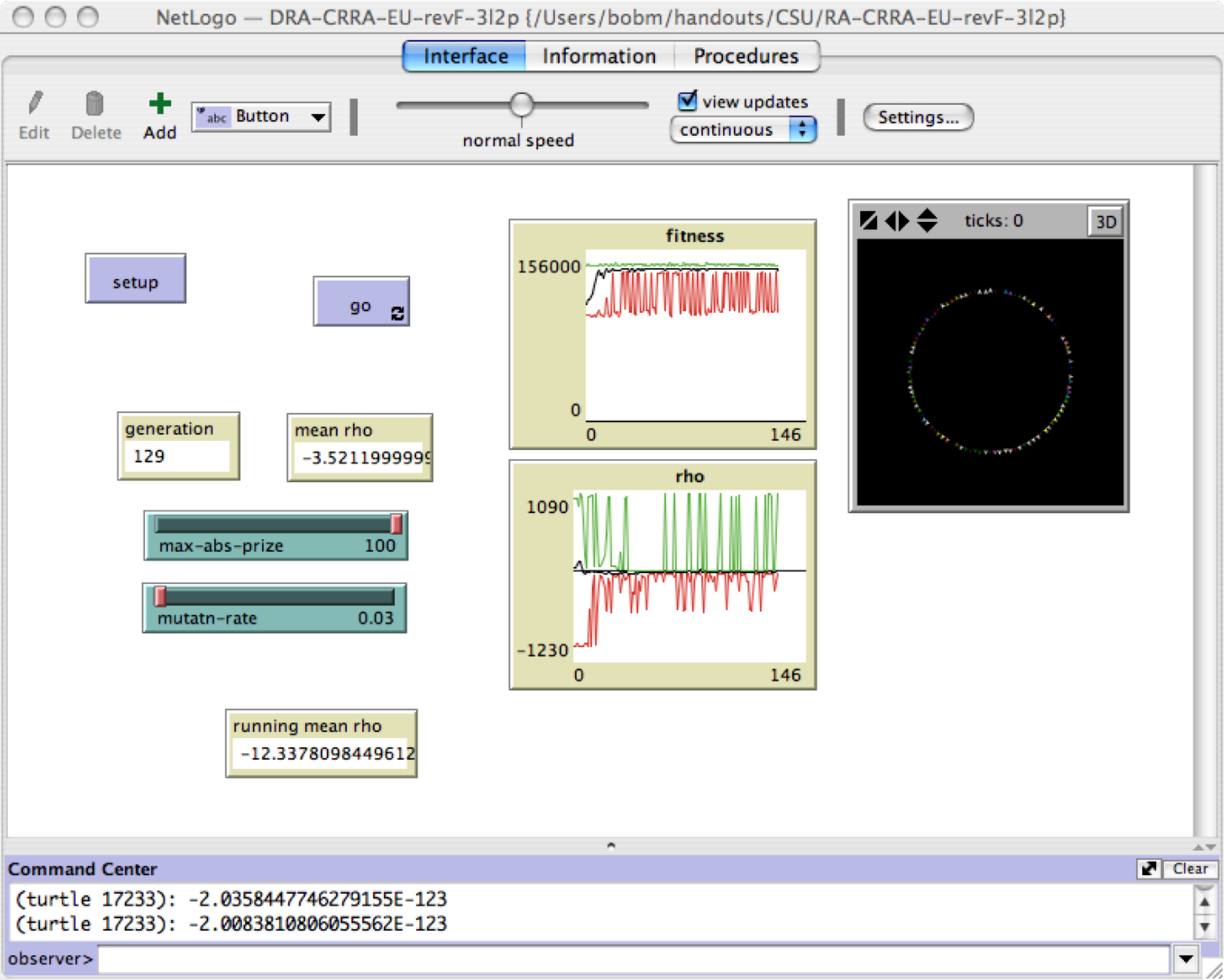
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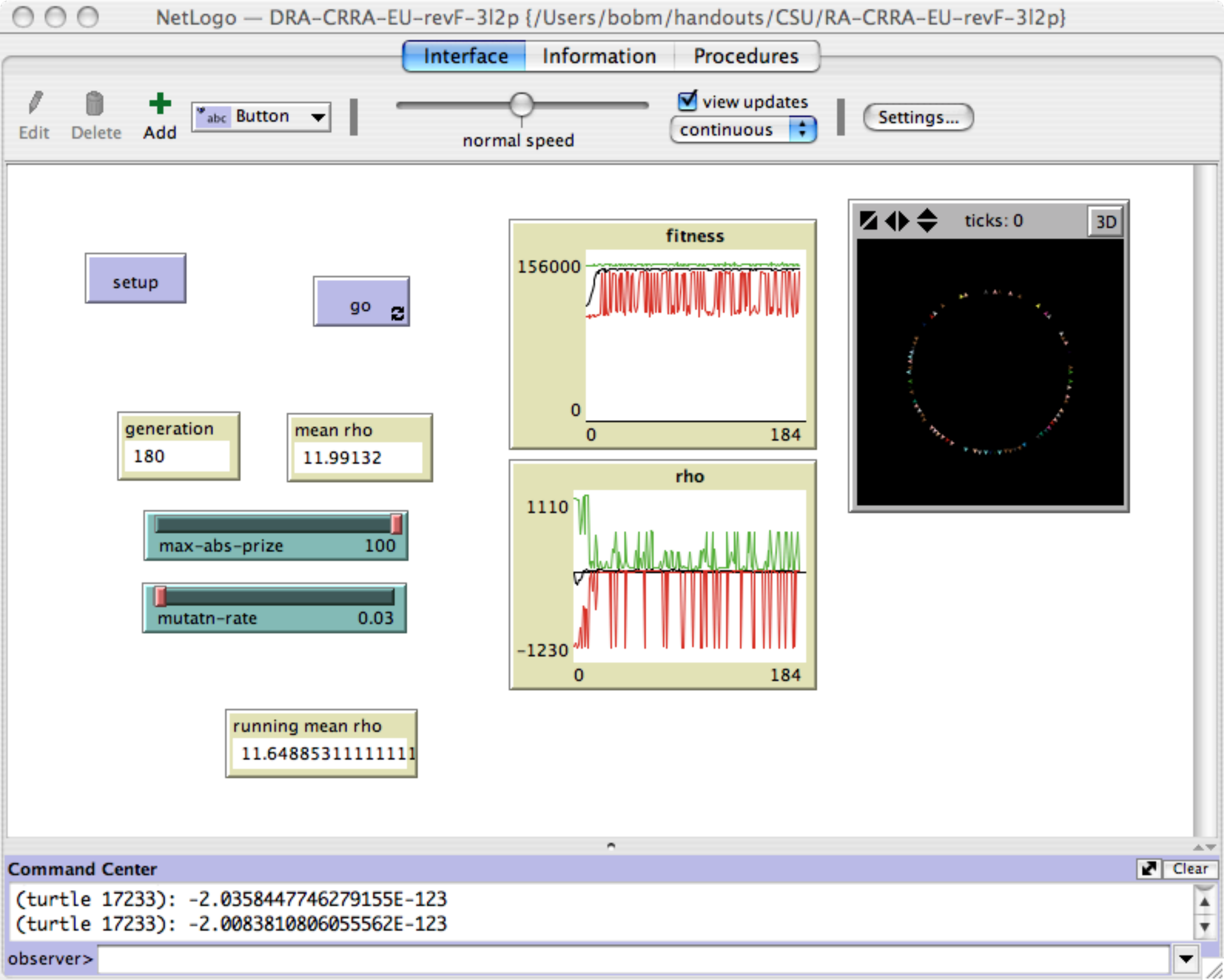
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4.3 Changing the Fitness Function

For the simulations in 4.1.(CARA) and 4.2 (CRRA), the GA's fitness function (or performance measure) is the arithmetic mean of the agent's cumulative winning (or losings) in the 1000 lottery choices:

$$fitness = \frac{1}{1000} \sum_t L_{kt}$$

where L_{kt} is the realisation of the the highest-expected-utility lottery k, chosen at period t.

We now use the geometric mean:

$$fitness = \prod_t (L_{kt})^{0.001}$$

The motivation behind this is that the log of the geometric mean of the expected value of a lottery equals its expected utility with a logarithmic utility function.

Does risk-averse emerge now?

See http://www.agsm.edu.au/bobm/teaching/SimSS/NetLogo4-models/RA-CARA-EU-GM-revC.html for a Java aplet and the NetLogo code of the CARA model with Geometric Mean.

and see
http://www.agsm.edu.au/bobm/teaching/SimSS/NetLogo4-models/DRA-CRRA-EU-GM-revH-3l2p.html **for a Java aplet and the NetLogo code of the CRRA model with Geometric Mean.**

Does risk aversion emerge?

This requires $\gamma > 0$ (for CARA) and $\rho > 0$ (for CRRA).

5. Discussion

Unlike the GA simulations of Szpiro (1997), we find that the best-performing CARA agents are risk-neutral, not risk averse. Because of the indirect way in which Szpiro modelled the risk profiles of his agents (unlike a referee's suggestion, footnote 3, Szpiro's model "only distinguishes between risk-averse automata and all others"), explanation of the contradictory results is not easy, but since our models allow any risk profile to emerge, we argue that they are more general than Szpiro's.

Should we be surprised that risk neutrality does better than risk aversion in CARA utility functions? Rabin (2000) suggests a reason why not, at least for small-stakes lotteries. He argues that von Neumann-Morgenstern expected-utility theory is inappropriate for reconciling actual human behaviour as revealed in risk attitudes over large stakes and small stakes. If there is risk aversion for small stakes, then expected-utility theory predicts wildly unrealistic risk aversion when the decision maker is faced with large stakes. Or risk aversion for large stakes must be accompanied by virtual risk neutrality for small stakes.

Rabin argues that *loss aversion* (Kahneman and Tversky 1979), rather than risk aversion, is a better (i.e. more realistic) explanation of how people actually behave when faced with risky decisions. This suggests possibilities for further simulations, although "loss aversion" suggests a prior conclusion.

But we do not appeal to empirical evidence or even to prior beliefs of what sort of risk profile is best. Whereas there has been much research into reconciling actual human decision making with theory (see Arthur 1991), we are interested in seeing what is the best (i.e. most profitable) risk profile for agents faced with risky choices.

And we find that for wealth-independent CARA utility functions (exponential) agents learn to become risk-neutral decision makers in order to maximise their returns when choosing among risky propositions. This is different from the risk-averse agents that Szpiro (1997) observed. But for wealth-dependent CRRA utility functions (CES) our agents often do learn to be slightly risk averse, as expected, but not always.

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