Modelling Heterogeneous Inputs

1. Introduction

How should we model heterogeneous inputs when there are non-linearities? For simplicity's sake, let us assume that there are two types of workers: "low" and "high" productivity. If the relationship among the numbers of workers of each type and their output is non-linear, that is, if the production does not function exhibit constant returns to scale (CRTS), then this problem is non-trivial. With a workforce made up of even just two types of labor, it turns out that there are many ways of modelling non-homogeneous labor. Which is best?

Let the two-input production function be denoted by:

$$Y = G(L, H),$$

where L is the amount of "low" productivity labor and H is the amount of "high" productivity labor, with positive marginal products with respect to the two inputs: [1]

$$G_i(L, H) > 0, \qquad \text{for all } L, H, \; i = L, H.$$

We consider three possible formulations of $G(.,.)$:

1. *Additively Separable* types.

$$Y = F(H) + c\,F(L), \quad 0 < c \leq 1. \tag{1}$$

   Output Y is the sum of a function $F(.)$ of the number of "highs," H, and a fraction c times the function of "lows," L, where F is a single-input function with strict monotonicity, and non-linear, in general.

2. *Linear Combination* of types.

1. We also assume that F and G are finite, non-negative, real-valued, and single-valued for all non-negative and finite inputs, and that they are continuous and twice-differentiable.

$$Y = h\,F(N) + c\,(1\text{–}h)\,F(N), \tag{2}$$

where h is the proportion of "highs", H/N. Output is a function of the total number of workers, $N \equiv H + L$, times a factor which is unity if all are "highs", and the fraction c if all are "lows".

3. *Effective Labor.*

$$Y = F(H + c\,L). \tag{3}$$

Output is a function of the amount of "effective labor," which is equal to N if all are "highs" and to the fraction c of N if all are "lows".

How shall we compare these formulations? If we want to model pairs of inputs as perfect substitutes up to a constant multiple, we shall see that with non-linear production technology, only the third formulation will suffice.

2. Equal Output or Equal Numbers

The Effective Labor formulation exhibits the property of *Equal Output*, in which the ratio of the numbers of "highs" alone to "lows" alone to produce equal output is constant and equal to the fraction c for any level of output:

$$G(N,\,0) = G(0,\,cN),\ 0 < c \le 1,\ \forall\,N. \tag{4}$$

The obvious alternative is that of *Equal Numbers*, in which the ratio of the maximum levels of output produced by an equal number of "lows" alone and "highs" alone is constant and less than one, whatever the number of homogeneous workers of each type:

$$G(N,\,0) = e\,G(0,N),\ 0 < e \le 1,\ \forall\,N. \tag{5}$$

The Linear Combination formulation exhibits this property.

It is easily shown that Equal Numbers and Equal Output properties are identical (with $c = e$) if and only if the production function G exhibits CRTS.

Lemma 1: *With $c = e$, the two properties—Equal Output and Equal Numbers—are identical if and only if the production technology exhibits CRTS.*

Proof: From equations (4) and (5), for the two properties to be identical requires $c = e$, and

$$G(N,\, 0) = G(0,\, cN) = c\, G(0,\, N),$$

which always holds with CRTS. With diminishing returns to scale (DRTS), $G(0, cN) > c\, G(0, N)$, with $0 < c \le 1$. With increasing returns to scale (IRTS), $G(0, cN) < c\, G(0, N)$. □

Corollary: The two formulations, Linear Combinations and Effective Labor, are identical if and only if the production technology exhibits CRTS.

Neither of these properties describes the behaviour of the production process with a heterogeneous mixture of "lows" and "highs" employed by the same firm; nothing is implied about concavity (Chambers 1988).

Assumption: To make the point more strongly, we model the two types of labor inputs, "high" and "low," as *perfect substitutes up to a constant multiple.*

Lemma 2: *With the assumption of perfect substitutability up to a multiple, the Equal Output property implies a homothetic production function, with parallel, linear isoquants.*

Proof: The assumption of perfect substitutability up to a multiple results in a family of linear isoquants, with the equation

$$L = -m\, H + \text{a constant}, \qquad m \ge 1.$$

A homothetic production function is yielded by a monotonically increasing transformation of a homogeneous production function, and can exhibit CRTS, IRTS, or DRTS. Homotheticity requires that $dm/dY = 0$, and this is satisfied with the assumption of Equal Output, since in that case, from equation (4), we see that the points $(N, 0)$ and $(0, cN)$ are the intercepts of each isoquant, with $m = 1/c$. □

It is convenient to parameterise the production function in terms of a measure of "effective labor", which is constant along each isoquant. With the assumptions of perfect substitutability up to a multiple, and of a homothetic production function, the isoquants are a family of parallel, straight lines, with the equation

$$L = -mH + \text{constant.}$$

Definition: Hence we define *effective labor,* $\hat{N}$, as

$$\hat{N} \equiv \frac{1}{m} L + H, \qquad m \geq 1.$$

We can now describe the production function of equation (3) with the equation

$$Y = G(L, H) = F(\hat{N}) = F\left(\frac{L}{m} + H\right). \tag{6}$$

When all workers are "high," $\hat{N} = H$, and when all workers are "low," $\hat{N} = L/m$. At any combination of "high" and "low" labor, the ratio of marginal products is constant, and given by

$$\frac{F_H}{F_L} = m \geq 1.$$

We have shown above in Lemma 2 that the Equal Output property implies homotheticity, given perfect substitutability up to a multiple. In this case $m = 1/c$, and effective labor is given by

$$\hat{N} = cL + H, \qquad 0 < c \leq 1.$$

What of the alternative property, Equal Numbers, described in equation (5)?

Lemma 3: *With the assumptions of perfect substitutability up to a multiple and a* *homothetic production function, the Equal Numbers property can only occur in the case of CRTS.*

Proof: The two assumptions allow us to parameterise the production function. From equations (5) and (6), we require that

$$F(\frac{N}{m}) = e\,F(N), \qquad \forall\, N.$$

Homotheticity implies that all expansion paths are linear, which implies that m is a constant. If $m = 1/e$, then this can only hold if the production function exhibits CRTS, (in which case the two formulations are identical, from Lemma 1). □

Corollary: Only in the case of CRTS can the Linear Combination of types production function occur with inputs perfectly substitutable up to a multiple.

3. The Effective Labor formulation of heterogeneous labor

We wish to examine the cases of CRTS, DRTS, and IRTS, with the scale independence of homothetic production functions. With the assumptions of homotheticity and perfect substitutability up to a multiple, we resolve the issue of how best to model heterogeneous productivity.

Definition: The *effectiveness factor,* $g(h)$, is defined as

$$g(h) \equiv \hat{N}/N, \qquad 0 < g(h) \leq 1,$$

the ratio of effective labor to all "high" productivity workers. Note that in our two-type example, $g(h) = h + (1-h)c$, so equation (2) becomes $Y = g(h)F(N)$, and equation (3) $Y = F(g(h)N)$.

Theorem: *Only the Effective Labor formulation exhibits perfect substitutability up to a multiple for heterogeneous inputs with non-linear production.*

Proof: From the Corollary of Lemma 3, non-linearity and perfect substitutability up to a multiple rule out the Linear Combinations of types formulation. Moreover, we can calculate the equations for the slopes of the isoquants of the three formulations: With Additively Separable types:

$$\left.\frac{dL}{dH}\right|_{\bar{Y}} = -\frac{1}{c}\,\frac{F'(H)}{F'(L)},$$

which, in general, is not constant.

With a Linear Combination of types:

$$\left.\frac{dL}{dH}\right|_{\bar{Y}} = -\,\frac{1-g\,(1-\gamma)}{c-g\,(1-\gamma)},$$

where $\gamma = N\,F'(N)/F(N)$, the ratio of marginal to average productivity at N. Only with CRTS will $\gamma = 1$, and the isoquants be linear.

With the Effective Labor formulation:

$$\left.\frac{dL}{dH}\right|_{\bar{Y}} = -\frac{1}{c}.$$

Only the third formulation exhibits perfect substitutability up to a multiple for the heterogeneous labor inputs, since only this formulation has linear isoquants with CRTS, IRTS, or DRTS. □

4. Many types of inputs

We can readily extend the formulations and proofs to the instance in which there are n types of heterogenous labor, x_i, $i = 1, \cdots, n$, and the total number of workers, $N = \sum x_i$.

1. The case of Additive Separability becomes:

$$Y = \sum_{i=1}^{n} \beta_i\,F(x_i),$$

where β_i is the ratio of the productivity of type i to the highest-productivity type 1, normalised setting β_1 equal to 1, and $\beta_i \leq 1$, $i = 2, \cdots, n$, without loss of generality.

2. The case of a Linear Combination of types becomes:

$$Y = \sum_{i=1}^{n} \alpha_i \, \beta_i \, F(N) = g \, F(N),$$

where the β_is are as above, and the α_is are the share of each type i in the total number (N), $\alpha_i = x_i / \sum x_i$, or x_i / N, and where g is the generalisation of $g(h)$, effectiveness factor, which becomes a function of $\{\alpha_i\}$, the distribution of shares of types, and the β_is:[2]

$$g = \sum_{i=1}^{n} \alpha_i \beta_i .$$

which allows us to write the effective labor as

$$\hat{N} = gN = \sum_{i=1}^{n} \alpha_i \beta_i N = \sum_{i=1}^{n} \beta_i x_i .$$

3. The Effective Labor formulation becomes:

$$Y = F\left(\sum_{i=1}^{n} \alpha_i \, \beta_i \, N \right) = F(\hat{N}).$$

5. References

Chambers, R.G., 1988, *Applied Production Analysis* (Cambridge University Press, Cambridge).

2. For continuous distributions $\alpha(i)$ and $\beta(i)$ over the set T of types, $g = \int_{i \in T} \alpha(i) \beta(i) \, di$.

Modelling Heterogeneous Inputs

Robert E. Marks

Graduate School of Business,
Stanford University.
(until end-August 1993)

Australian Graduate School of Management,
University of New South Wales.

r.marks@unsw.edu.au

JEL classification number: D24

ABSTRACT: If heterogeneous factor inputs are perfect substitutes up to a multiple, then the production function which exhibits the "Equal Output" property must be homothetic, and the natural formulation to use with non-linearities.

Acknowledgements: The author wishes to thank Andre Perold for his advice. The research was partly supported by the Australian Research Council and the Graduate School of Business, Stanford.

The author's present address (until mid-August 1993) is the Graduate School of Business, Stanford University, Stanford, CA 94305-5015, USA; thereafter, the Australian Graduate School of Management, University of New South Wales, Kensington, NSW 2033, Australia.